Exercises for Section 5.6

1. If $I := [a, b]$ is an interval and $f : I \to \mathbb{R}$ is an increasing function, then the point a [respectively, b] is an absolute minimum [respectively, maximum] point for f on I. If f is strictly increasing, then a is the only absolute minimum point for f on I.
2. If f and g are increasing functions on an interval $I \subseteq \mathbb{R}$, show that $f + g$ is an increasing function on I. If f is also strictly increasing on I, then $f + g$ is strictly increasing on I.
3. Show that both $f(x) := x$ and $g(x) := x - 1$ are strictly increasing on $I := [0, 1]$, but that their product fg is not increasing on I.
4. Show that if f and g are positive increasing functions on an interval I, then their product fg is increasing on I.
5. Show that if $I := [a, b]$ and $f : I \to \mathbb{R}$ is increasing on I, then f is continuous at a if and only if $f(a) = \inf\{f(x) : x \in (a, b]\}$.
6. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Show that f is continuous at c if and only if there exists a sequence (x_n) in I such that $x_n < c$ for $n = 1, 3, 5, \ldots$; $x_n > c$ for $n = 2, 4, 6, \ldots$; and such that $c = \lim(x_n)$ and $f(c) = \lim(f(x_n))$.
7. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. If c is not an endpoint of I, show that the jump $j_f(c)$ of f at c is given by $\inf\{f(y) - f(x) : x < c < y,\ x, y \in I\}$.
8. Let f, g be strictly increasing on an interval $I \subseteq \mathbb{R}$ and let $f(x) > g(x)$ for all $x \in I$. If $y \in f(I) \cap g(I)$, show that $f^{-1}(y) < g^{-1}(y)$. [*Hint*: First interpret this statement geometrically.]
9. Let $I := [0, 1]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) := x$ for x rational, and $f(x) := 1 - x$ for x irrational. Show that f is injective on I and that $f(f(x)) = x$ for all $x \in I$. (Hence f is its own inverse function!) Show that f is continuous only at the point $x = \frac{1}{2}$.
10. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If f has an absolute maximum [respectively, minimum] at an interior point c of I, show that f is not injective on I.
11. Let $f(x) := x$ for $x \in [0, 1]$, and $f(x) := 1 + x$ for $x \in (1, 2]$. Show that f and f^{-1} are strictly increasing. Are f and f^{-1} continuous at every point?
12. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that does not take on any of its values twice and with $f(0) < f(1)$. Show that f is strictly increasing on $[0, 1]$.
13. Let $h : [0, 1] \to \mathbb{R}$ be a function that takes on each of its values exactly twice. Show that h cannot be continuous at every point. [*Hint*: If $c_1 < c_2$ are the points where h attains its supremum, show that $c_1 = 0$, $c_2 = 1$. Now examine the points where h attains its infimum.]
14. Let $x \in \mathbb{R}$, $x > 0$. Show that if $m, p \in \mathbb{Z}$, $n, q \in \mathbb{N}$, and $mq = np$, then $\left(x^{1/n}\right)^m = \left(x^{1/q}\right)^p$.
15. If $x \in \mathbb{R}$, $x > 0$, and if $r, s \in \mathbb{Q}$, show that $x^r x^s = x^{r+s} = x^s x^r$ and $(x^r)^s = x^{rs} = (x^s)^r$.